The Exciting Physics of an Excited Universe

Marton Trencseni, mtrencseni@gmail.com

Abstract

Based on the book *Compact Stars* by N. K. Glendenning a general relativistic treatment of pressure and energy in compact stars is given, followed by simple models of neutron and quark stars. Hypothetical strange quark matter and strange stars are introduced highlighting the importance of sub-millisecond pulsar detections.

1 Introduction

Compact stars, including neutron stars, white dwarfs and hypothetical strange quark stars are the remnants of luminous stars like our own. They are characterized by their small radius, on the order of 10 km (1000 km for white dwarfs) with masses on the order of one solar mass. This makes them 14 orders of magnitude denser than earth, whose average density is only $7g/cm^3$.

Like all stars, these compact remnants are allowed to rotate around their axis, with frequencies of up to $1000Hz$ and possibly higher. This rotation causes a distinctive astrophysical radio signal which can be measured: from an observational standpoint, these objects are known as pulsars, and in fact several thousand are known to exists in the Milky Way.

The employed theoretical framework is unlike the study of luminous stars, since the latter are well described by Newtonian and thermonuclear physics reproducible in reactors and accelerators, while the study of compact objects lies at the intersection of the major branches of modern theoretical physics, including Einstein's General Theory or Relativity, particle physics and the study of statistical physics (eg. Fermi-Dirac statistics and phase transitions).

Neutron stars, our starting point in this essay, are gravitationally bound with Fermi-Dirac statistics and short-range nuclear repulsion providing the outward pressure balancing gravity. The constituents, neutrons, protons and electons are degenerate meaning they occupy the lowest possible energy states. The existense of neutron stars is widely accepted since it requires no new physics, and provides a natural explanation for pulsars.

Strange quark stars are hypothetical objects qualitatively different from other compact stars such as neutron stars. First, they are composed of strange quark matter, a hypothetical form of matter consisting of equal part u, d and s quarks. This phase must be of lower energy than hadronic matter for a strange star to be stable to quark fusion. Second, unlike neutrons stars which are held together by gravity, strange stars are bound by the strong interaction. This creates a unique observational limit: gravitationally bound compact objects cannot have pulsar periods less than $0.3msec$ and pulsars with periods less than $1.0msec$ would already be strong evidence. The pulsar period is of course radius and mass dependent, but is bound from below by a neutron star just on the verge of forming a black hole singularity. As strange stars are not bound by gravity, they are free to occupy the *sub-millisecond pulsar* region.

2 Schwarzschild solution

Neutron stars are spherically symmetric relativistic objects decribed by Einstein's General Theory of Relativity. We look for static solutions and work with the metric of the from:

$$ds^2 = e^{2\nu(r)}dt^2 - e^{2\lambda(r)}dr^2 - r^2(d\theta^2 - sin^2\theta d\phi^2)$$

or in matrix form:

$$g_{\mu\nu} = \begin{pmatrix} e^{2\nu(r)} & 0 & 0 & 0 \\ 0 & -e^{2\lambda(r)} & 0 & 0 \\ 0 & 0 & -r^2 & 0 \\ 0 & 0 & 0 & -r^2 sin^2\theta \end{pmatrix}$$

For metrics of this simple form, the inverse is simply given by $g^{\mu\nu} = 1/g_{\mu\nu}$. With the metric above, one can go about the usual business of deriving the Christoffel symbols $\Gamma^\sigma_{\mu\nu}$, the Ricci-tensor $R_{\mu\nu}$, the Ricci scalar R, and finally the Einstein tensor $G_{\mu\nu}$ to arrive at the Eintein-equations.

We follow the standard General Relativity text of Carrol to define the Christoffel symbols as[1]

$$\Gamma^\sigma_{\mu\nu} = \frac{1}{2} g^{\sigma\rho}(\partial_\mu g_{\nu\rho} + \partial_\nu g_{\rho\mu} - \partial_\rho g_{\mu\nu})$$

A standard computer algebra system like Mathematica quickly computes the explicit form of Γ matrices:

$$\Gamma^0_{\mu\nu} = \begin{pmatrix} 0 & \nu' & 0 & 0 \\ \nu' & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

$$\Gamma^1_{\mu\nu} = \begin{pmatrix} e^{-2\lambda+2\nu}\nu' & 0 & 0 & 0 \\ 0 & \lambda' & 0 & 0 \\ 0 & 0 & -e^{-2\lambda}r & 0 \\ 0 & 0 & 0 & -e^{-2\lambda} r sin^2\theta \end{pmatrix}$$

$$\Gamma^2_{\mu\nu} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 1/r & 0 \\ 0 & 1/r & 0 & 0 \\ 0 & 0 & 0 & -cos\theta sin\theta \end{pmatrix}$$

$$\Gamma^3_{\mu\nu} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1/r \\ 0 & 0 & 0 & 1/tan\theta \\ 0 & 1/r & 1/tan\theta & 0 \end{pmatrix}$$

Bypassing Riemann, the Ricci tensor is defined by:

$$R_{\mu\nu} = \partial_\kappa \Gamma^\kappa_{\nu\mu} - \partial_\nu \Gamma^\kappa_{\kappa\mu} + \Gamma^\kappa_{\kappa\eta}\Gamma^\eta_{\nu\mu} - \Gamma^\kappa_{\nu\eta}\Gamma^\eta_{\kappa\mu}$$

which for our metric is diagonal and reads:

$$R_{00} = [e^{-2\lambda+2\nu}((2 - r\lambda')\nu' + r\nu'^2 + r\nu'')]/r$$

[1]Caution: Glendenning uses a different convention! The physical end-result is of course the same.

$$R_{11} = [\lambda'(2 + r\nu') - r(\nu'^2 + \nu'')]/r$$
$$R_{22} = e^{-2\lambda}(-1 + e^{2\lambda} + r\lambda' - r\nu')$$
$$R_{33} = sin^2\theta R_{22} = e^{-2\lambda} sin^2\theta(-1 + e^{2\lambda} + r\lambda' - r\nu')$$

The Ricci tensor $R = g^{\mu\nu} R_{\mu\nu}$ turns out to be:

$$R = [e^{-2\lambda}(2 - 2e^{2\lambda} + 4r\nu' + 2r^2\nu'^2 - 2r\lambda'(2 + r\nu') + 2r^2\nu'')]/r^2$$

We are now in a position to calculate the Einstein tensor defined by:

$$G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu}$$

which reads

$$G_{00} = [e^{-2\lambda+2\nu}(-1 + e^{2\lambda} + 2r\lambda')]/r^2$$
$$G_{11} = [1 - e^{2\lambda} + 2r\nu']/r^2$$
$$G_{22} = e^{-2\lambda} r(\nu' + r\nu'^2 - \lambda'(1 + r\nu') + r\nu'')$$
$$G_{33} = e^{-2\lambda} r sin^2\theta(\nu' + r\nu'^2 - \lambda'(1 + r\nu') + r\nu'')$$

The Einstein field equation is

$$G_{\mu\nu} = 8\pi G T_{\mu\nu}$$

where $T_{\mu\nu}$ is the stress-energy tensor of matter. This is a system of 4 coupled partial differential equations which determine the structure of spacetime ($\nu(r)$ and $\lambda(r)$) and that of matter ($T_{\mu\nu}$ through its components energy density $\epsilon(r)$ and pressure $p(r)$). Switching to the up-down convention (using $A^\mu_\nu = g^{\mu\kappa} A_{\kappa\nu}$ for any tensor $A_{\mu\nu}$) the stress-energy tensor has the particularly simple form $T^0_0 = \epsilon(r)$ and $T^i_i = -p(r)$.

First we concentrate on the region outside the star, where $T^\mu_\nu = 0$. Then the equation $G^0_0 - G^1_1 = 0$ simplifies to:

$$\lambda' + \nu' = 0$$

For large r, the spacetime must be asymptotically Minkowski, ie. $\lambda(r) \to 0$ and $\nu(r) \to 0$ in the $r \to \infty$ limit. Therefore

$$\lambda + \nu = 0$$

$G^2_2 = 0$ simplifies to:

$$2\nu' + 2r\nu'^2 + r\nu'' = 0$$

Integrating the above equation yields:

$$\nu(r) = C_2 + \frac{1}{2} log(rC_1 - 2] - \frac{1}{2} log r$$

Trivial applications of logarithm rules yield:

$$e^{2\nu(r)} = K_1(1 - \frac{2K_2}{r})$$

Noting the role of $e^{2\nu(r)}$ in the metric expression the constant of integration K_1 may always be chosen to be 1 by redefining the time coordinate. Taking into account the Newtonian weak field limit, $K_2 = GM$ and we identify M with the mass of the compact star. The metric in the outer vacuum region thus reads

$$ds^2 = (1 - \frac{2GM}{r})dt^2 - (1 - \frac{2GM}{r})^{-1}dr^2 - r^2(d\theta^2 - sin^2\theta d\phi^2)$$

This is the so-called *Schwarzschild solution* and holds for any static, spherically symmetric object of mass M including neutron stars, strange stars and black holes. It is singular at the Schwarzschild radius $r_S = 2GM$, but this is a superficial singularity, not a true spacetime singularity — one can get rid of it by redefining coordinates. Nevertheless, it has a special significance for black holes, where it is the event horizon, from which no particle or information can escape (classically).

3 Oppenheimer-Volkoff equations

We now focus our attention to the inside of the star, where $T^\mu_\nu \neq 0$. The Einstein equations in up-down form read (with $k = 8\pi G$)[2]:

$$\begin{aligned}
&r^2 G^0_0 = e^{-2\lambda}(-1 + e^{2\lambda} + 2r\lambda') = r^2 k T^0_0 = r^2 k \epsilon(r) \\
&r^2 G^1_1 = e^{-2\lambda}(-1 + e^{2\lambda} - 2r\nu') = r^2 k T^1_1 = -r^2 k p(r) \\
&r G^2_2 = -e^{-2\lambda}(\nu' + r\nu'^2 - \lambda'(1 + r\nu') + r\nu'') = rkT^2_2 = -rkp(r) \\
&G^3_3 = G^2_2
\end{aligned}$$

We note that $r^2 G^0_0 = \frac{d}{dr}\{r(1 - e^{-2\lambda})\}$, which used in the zeroth Einstein equation yields $e^{-2\lambda(r)} = 1 - \frac{k}{r}\int_0^r r^2\epsilon(r')dr'$. Defining

$$M(r) = 4\pi \int_0^r \epsilon(r')r'^2 dr'$$

we see the connection with the vacuum form of the metric:

$$e^{-2\lambda(r)} = 1 - \frac{2GM(r)}{r}$$

Performing further transformations on the three independent Einstein equations above it is possible to eliminate the metric functions $\nu(r)$ and $\lambda(r)$ and arrive at the differential form:

$$\frac{dp}{dr} = -G\frac{[p(r) + \epsilon(r)][M(r) + 4\pi r^3 p(r)]}{r[r - 2GM(r)]}$$

This is the famous *Oppenheimer-Volkoff equation* for a static, spherically symmetric star. It is remarkable because it is model independent — it is true for any matter, be it hadronic or quarkonic, hot or cold, degenerate or excited. Given the Oppenheimer-Volkoff equation, a practinioner has to plug in an *equation of state* of the form:

$$p(r) = p(\epsilon(r))$$

specify the inital conditions $\epsilon(r = 0) = \epsilon_0$ and perform the integration. In practice, the integration must usually be performed numerically.

[2]Glendenning uses $k = -8\pi G$ to compensate for the non-conventional definition of the Christoffel symbols.

The Oppenheimer-Volkoff equation describes a compact star in hydrostatic equilibrium, but has nothing to say about *stability* as it may correspond to a minimum or maximum. There is in fact a broad $\epsilon_0 - M$ region spanning four orders of magnitude between stable white dwarfs and stable neutron stars that is unstable. The condition for stability is

$$\frac{\partial M}{\partial \epsilon_0} > 0$$

with instability setting in at equality. This is intuitive, as it says that the equilibrium star with increased central energy density lies at higher overall mass, hence the perturbed star has a mass deficit, so pressure will drive to return the star to its original configuration.[3] For polytropic equations of state $p = \epsilon^\gamma$ this condition translates to $\gamma > 4/3$.

The factors on the right are all positive ($p(r)$, $\epsilon(r)$ and $M(r)$ are positive, [r - 2 G M(r)] because of the form of the metric), which means that $p(r)$ is a monotonically decreasing function: pressure is highest at the center of the star and decreases outward, as we would expect. This is also in accord with the condition for a static solution: pressure balances gravity, and with increasing r less weight must be supported. The edge of the star is defined to be where $p(r_{edge}) = 0$, also determined by the Oppenheimer-Volkoff equation.

We note that $M = M(r_{edge})$ is larger than the combined mass of the constituents of the star if they were dispersed to infinity, since it is mass-energy which includes gravitational binding. The gravitational binding energy is of the order of 100 MeV per nucleon, compared to 10 MeV for nucleons bound by the strong force. This is explained by strong force being a short range force, with nucleons only interacting with their nearest neighbours, while gravity is long range, which means that all nucleons in the star attract.

Pressure appears on the right side of the Oppenheimer-Volkoff equations together with mass-energy *compressing* matter further — a prediction unique to general relativity. This means that one cannot make arbitrarily compact stars by increasing the pressure, as increased pressure leads to a steeper pressure gradient and eventually the star's radius will fall below the Schwarzschild radius r_S and collapse to a black hole.

Mass and radius have a peculiar relationship: as mass increases, the gravitational pull becomes stronger, and the star becomes more compact. As matter is taken away, the gravitational pull decreases, the star "relaxes" and the radius increases. This is in contrast to *saturated* objects such as hypothetical strange quark stars which are not bound by gravity and their density cannot increase further.

Each possible equation of state $p = p(\epsilon)$ defines a unique sequence of compact stars according to the Oppenheimer-Volkoff equation[4], each member of the sequence parameterized by the central energy density $\epsilon(r = 0) = \epsilon_0$ (or equivalently central pressure). White dwarfs are the lowest density sequence, with $6 < log\epsilon_0 < 10$ $[g/cm^3]$. White dwarfs are supported by the Fermi pressure of low temperature, Fermi-degenerate electron gas. Above $\sim 10^{10}$ g/cm^3 central energy it becomes possible for electrons to be captured by protons in inverse beta decay, therefore such an object would not have the required supporting pressure.

[3]See Glendenning p. 131.

[4]See Glendenning Figure 3.6

4 Neutron stars

Chadwick discovered the neutron in 1932, and already in 1933-34 Baade and Zwicky hypothesized the existence of neutron stars while looking for the driving force behind supernova explosions.

Neutron stars are more compact than white dwarfs, with central densities in the range $14 < log\epsilon_0 < 16$ $[g/cm^3]$. Neutron stars, like white dwarfs are held together by gravity (nuclei above $A > 250$ are unstable), with the strong nuclear force actually working against gravity, providing outward pressure.

It should be noted that when deriving the equation of state of neutron stars, we are free to employ Special Relativity and disregard General Relativity, even though in the end the Oppenheimer-Volkoff equation is about gravity. This is simply because the processes determining the equation of state are of a short timescale, hence short distances so that the gravitational potential $\sim g_{00}$ (or curvature of spacetime) does not change appreciably.

We now derive a useful property of neutron stars, the "constancy" of the chemical potential. We note that subtracting the first from the zeroth Einstein equation and using the already derived form of $e^{-2\lambda(r)}$ and the definition of $M(r)$ one can deduce:

$$\frac{d\nu}{dr} = G\frac{GM(r) + 4\pi r^3 p(r)}{r[r - 2GM(r)]}$$

which together with the Oppenheimer-Volkoff equations yields:

$$\frac{d\nu}{dr} = -\frac{1}{\epsilon(r) + p(r)}\frac{dp}{dr}$$

Then, using the definition of pressure at contstant entropy $p = \partial E/\partial V$ and $\epsilon = E/V$ and $\rho = A/V$ we see (where $\mu = d\epsilon/d\rho$):

$$p = -\frac{\partial(\epsilon/\rho)}{\partial(1/\rho)} = \rho\mu - \epsilon$$

Taking the derivative:

$$\frac{dp}{d\rho} = \rho\frac{d\mu}{d\rho}$$

Multiplying both sides by $d\rho/dr$ and formally canceling we arrive at:

$$\frac{dp}{dr}\frac{1}{\rho} = \frac{d\mu}{dr}$$

Using the above equation and $\epsilon + p = \rho\mu$ in the formula for $d\nu/dr$ we arrive at:

$$-\int_{r_0}^{r_1} \nu'(r)dr = \int_{r_0}^{r_1} \frac{\mu'(r)}{\mu(r)}dr$$

which holds for $r_0 < r_1 < R$. Integrating:

$$\mu(r_0)e^{\nu(r_0} = \mu(r_1)e^{\nu(r_1)} = const$$

We see that this peculiar compbination of the chemical potential $\mu(r)$ and $e^\nu = \sqrt{g_{00}}$ is constant. Concentrating at the edge of the star we can deduce the value of the constant: here $e^\nu = (1 - \frac{2GM}{R})^{1/2}$ and $\mu(p = 0) = \mu_{Fe} = 930$ MeV, since solid iron is the lowest equilibrium energy state for hadronic matter. Incidentally, this also implies that neutron stars carry an iron shell of hadrons.

We continue by calculating the makeup of matter in neutron stars by imposing *chemical equilibrium of weak interaction channels* and *charge neutrality.* We assume that the electrons, neutrons and protons making up the star are described by degenerate Fermi-gas, meaning they occupy the lowest possible energy states. This is true if the temperature of the compact star is less than the mass of the particles in Kelvins). For electrons $m_e \sim 6 \times 10^9 K$, larger than the typical temperature of a neutron star in equilibrium ($10^6 - 10^8$ K). Working in momentum space, we use the following formulas (where $\gamma = 2$ is the spin degeneracy):

$$\epsilon = \frac{\gamma}{2\pi^2} \int_0^k \sqrt{k^2 + m^2} k^2 dk$$

$$p = \frac{\gamma}{6\pi^2} \int_0^k \frac{k^2}{\sqrt{k^2 + m^2}} k^2 dk$$

$$\rho = \frac{\gamma}{2\pi^2} \int_0^k k^2 dk$$

The peculiar form of p ensures that the thermodynamic relationship $p = \rho\mu - \epsilon$ holds. The equations can be integrated using standard integrals and yields (where $\mu = \mu_F = \sqrt{k_F^2 + m^2}$):

$$\epsilon = \frac{1}{4\pi^2}[\mu k(\mu^2 - \frac{1}{2}m^2) - \frac{1}{2}m^4 log\frac{\mu + k}{m}]$$

$$p = \frac{1}{12\pi^2}[\mu k(\mu^2 - \frac{5}{2}m^2) + \frac{3}{2}m^4 log\frac{\mu + k}{m}]$$

$$\rho = \frac{k^3}{3\pi^2}$$

We seek to minimize the total energy density of the neutron star $\epsilon = \epsilon_p + \epsilon_n + \epsilon_e$ at fixed baryon density $\rho = \rho_p + \rho_n$ while mainting charge neutrality $\rho_p = \rho_e$. Employing the method of Lagrange multipliers, we define

$$F = \epsilon + \alpha(\rho - \rho_n - \rho_p) + \beta(\rho_p - \rho_e)$$

The α part fixes baryon density while the β part ensures charge neutrality (we have one Lagrange multiplier for each conservation law). We require (keeping ρ constant):

$$\frac{\partial F}{\partial \rho_p} = \frac{\partial F}{\partial \rho_n} = \frac{\partial F}{\partial \rho_e} = 0$$

which, written out is:

$$\frac{\partial \epsilon}{\partial \rho_p} - \alpha + \beta = \frac{\partial \epsilon}{\partial \rho_n} - \alpha = \frac{\partial \epsilon}{\partial \rho_e} - \beta = 0$$

Using $\partial\epsilon/\partial\rho = (\partial\epsilon/\partial k)(\partial k/\partial\rho) = \sqrt{k^2 + m^2}$ for all particle species we arrive at:

$$\alpha = \mu_n \qquad \alpha + \beta = \mu_p \qquad -\beta = \mu_e \qquad \rightarrow \qquad \mu_n = \mu_p + \mu_e$$

We are almost done. Using the formula $\rho = k^3/(3\pi^2)$ we write for the baryon density $\rho_b = (k_p^3 + k_n^3)/(3\pi^2)$ and we write charge neutrality as $k_p = k_e$ (protons and electrons have to be equally degenerate, ie. their Fermi momentum must be equal).

First, take the $k_p = k_e = 0$ case: using the equality between chemical potentials, this translates to $\mu_n = \sqrt{k_n^2 + m_n^2} = m_p + m_e$. Plugging in the actual mass values we obtain $k_n^2 < 0$, a contradiction, hence $k_p = k_e > 0$ always. In other words, there are *always* some protons (and electrons) in a neutron star.

Second, take $k_n = 0$: repeating the same substitutions, we get $k_p^2 \sim 1.4$ MeV^2 (or $\epsilon \sim 10^7$ g/cm^3). Below this momentum there are no neutrons present, the star is an equal mixture of degenerate protons and electrons.

Finally, we write out the original equation $\mu_n = \mu_p + \mu_e$ using the formulas for the chemical potential and the equality for the baryon density to obtain (using $k_p = k_e$):

$$[(3\pi^2\rho_b - k_p^3)^{2/3} + m_n^2]^{1/2} = (k_p^2 + m_p^2)^{1/2} + (k_p^2 + m_e^2)^{1/2}$$

In the relativistic regime, where mass is negligible the equation simplifies to:

$$\rho_b = \frac{9k_p^3}{3\pi^2}$$

Taking into account $\rho_p = k_p^3/3\pi^2$ one arrives at the answer $\rho_p/\rho_b = 1/9$. This is an upper limit, meaning a neutron star always consists of at least 8/9 part neutrons.

In the derivation we have assumed beta-equilibrium (the freezing out of weak interaction channels), charge neutrality and have assumed that pressure arises from the Fermi degeneracy (Pauli principle). Further possible improvements include taking into account the nuclear isospin preference for $k_p = k_n$, additional baryon species, strong interactions and phase transitions. A more realistic model of neutron stars includes additional types of particles such as hyperons (baryons containing one or more s quark, but no c or b quarks)[5].

5 Pulsars

Pulsars are astrophysical objects that blink on an off at constant frequency. Pulsars are spinning neutron stars with jets of particles moving at relativistic speeds out from their magnetic poles, with the magnetic axis not aligned with the spin axis, resulting in the period signal as its light sweeps the Earth's direction, like a lighthouse. The angular width of the electromagnetic radiation is typically ~ 10 degrees.

Roughly ~ 1000 pulsars have been found to date, with frequencies ranging from 1.4 msec to 8500 msec. All observed pulsars like within the Milky Way with the exception of two, which lie in the Large and Small Magellanic Clouds. An optically visible neutron star RX J185635-3754 has been observed[6] by the HST with surface temperature of 7×10^5 K and no larger than $r \sim 14$ km.

The pulsation of pulsars is intepreted as *rotation* and not vibration of a magnetized star. Flux conservation of a collapsing star ensures that the neutron star carries a large magnetic field and provides the radiating mechanism: a rotating magnetic dipole with energy output $\sim 10^{44}$ MeV/s. It is believed that vibrational modes diminish over time, whereas rotational frequency does not.

[5] Such a more realistic model is given in Glendenning p. 233.

[6] See http://hubblesite.org/newscenter/archive/releases/2000/35/image/a/

6 Strange quark stars

Quark stars are composed of, in whole or in part (hybrid stars), quark matter. This phase of matter is the result of asymptotic freedom, the property of the strong interaction that at short distances the force becomes arbitrarily weak. In this phase, nucleons disintegrate and quarks are free to move about, forming deconfined colorless quark matter. It is possible that the Universe passed through such a phase shortly after the Big Bang, before nucleons formed.

The most interesting aspect of quark matter is the possibility that it is the true ground state of the strong interaction, with quarks confined to nucleons being a meta-stable state with a very long half-time. If this hypothesis were true, we would in effect be living in an Excited Universe, ourselves (and all hadronic matter) also being meta-stable excitations.

The question of the true ground state is undecided, as QCD and lattice simulations are of insufficient precision to yield a conclusive answer. Here we use the 1974 MIT bag model, a simpler model than more precise QFTs. The vacuum of QCD in this model contains no quarks, and then this vacuum must be expelled, which costs energy. The per volume energy cost of expelling the vacuum is the bag constant B [MeV], whose optimum value for our calculations is $B = 145$ MeV. The bag energy of quarks in a volume V is therefore $E_{bag} = BV$. The quarks are modeled as a Fermi gas, hence the other contribution is kinetic energy. The expressions of pressure, energy density, baryon density and entropy are similar as before (where $E_f = \sqrt{m_f^2 + k^2}$, f denotes quark flavor):

$$p = \sum_f \frac{1}{3}\frac{\gamma_f}{2\pi^2}\int_0^\infty k\frac{\partial E_f}{\partial k}[n(k,\mu_f) + n(k,-\mu_f)]k^2 dk - B$$

$$\epsilon = \sum_f \frac{\gamma_f}{2\pi^2}\int_0^\infty E_f[n(k,\mu_f) + n(k,-\mu_f)]k^2 dk + B$$

$$\rho = \sum_f \frac{1}{3}\frac{\gamma_f}{2\pi^2}\int_0^\infty [n(k,\mu_f) - n(k,-\mu_f)]k^2 dk$$

$$s = (\frac{\partial p}{\partial T})_{V,\mu_f}$$

Note the presence of the bag constant in the energy density and pressure and the negative sign in ρ (anti-particles). $\gamma_f = 2_{spin} \times 3_{color} = 6$ is the degeneracy, $n(k,\mu_f)$ is the Fermi-Dirac distribution function:

$$n(k,\mu_f) = \frac{1}{1 + exp[(E - \mu_f)T]}$$

Solving these equations is hard in general, but possible in some limiting cases: if $m_q = 0$ one obtains the explicit form $p(T,\mu_f), \rho(T,\mu_f), s(T,\mu_f)$ and $\epsilon(p) = 3p + 4B$ is the equation of state. Similar calculations[7] also yield the same functions for massles Bose-Einstein gas of gluons with equation of state $\epsilon = 3p$ (no bag model). For a non-interacting gas containing both quarks and gluons we simply add the contributions.

At this point, we have to specify the chemical equilibrium of quark matter (the formulas above contain μ_f). On the timescale of the star, strangeness is not conserved, only charge and baryon number. Examining the quantum numbers of u, d, s and c quarks we arrive at the equations:

[7]See Glendenning p. 326.

$$\mu_d = \mu_u + \mu_e \qquad \mu_s = \mu_d \qquad \mu_c = \mu_u$$

and charge conservation (quark species, electron and muon, the factor 3 comes from quark color degeneracy):

$$q = \sum_f q_f \frac{k_f^3}{\pi^2} - \frac{k_e^3 + k_\mu^3}{3\pi^2}$$

With these considerations the chemical potentials can be determined at fixed baryon density, which in turns means that the Oppenheimer-Volkoff equation can be solved.

In the bag model, the parameter B determines whether strange quark matter is stable with respect to hadronic matter or not. The critical value of B is one where the binding energy per baryon is slightly less than the mass of the nucleon $\sim$ 928 MeV, at $B^{1/4} = 154.5$ MeV for $m_s = 150$MeV. Decreasing B increases stability; when $B = 145$ MeV is reached nonstrange quark matter (containing only u and d quarks) becomes stable, incompatible with the presence of nucleons. Is the hypothetical stability of strange quark matter not incompatible with the presence of nucleons all around us? No, because in order to decay several strange quarks would have to be created in highly unlikely parallel processes, as strange quark matter consists of equal part u, d, and s quark to minimize the Fermi pressure.

Nevertheless, if the hypothesis is true, some compact stars may be made up of strange quark matter. How can we differentiate these hypothetical *strange stars* from neutron stars? It turns out that neutron and strange stars have about the same radius, but neutron stars are gravitationally bound, while saturated strange stars are held together by the strong interaction, and can *rotate faster* than the equivalent (in terms of M, R) neutron star. Thus, if we find an object which exceeds this limiting angular momentum, we can be fairly certain it is a strange star, which would validate the strange matter hypothesis, and would be a fundamental discovery. For example, for a typical neutron star of $1.44 M_\odot$ the limiting period is 0.3 msec. However, there is no physical requirement for neutron stars to maximize their rotational speed, and in fact no sub-millisecond pulsars are currently known. The discovery of one would be a strong hint at the stability of quark matter, with a 0.3 msec (or below) candidate being "conclusive".

References

[1] N. K. Glendenning, *Compact Stars*, 1997, Springer-Verlag, New York.

[2] S. M. Carroll, *An Introduction to Spacetime and Geometry*, 2004, Addison Wesley, San Francisco.